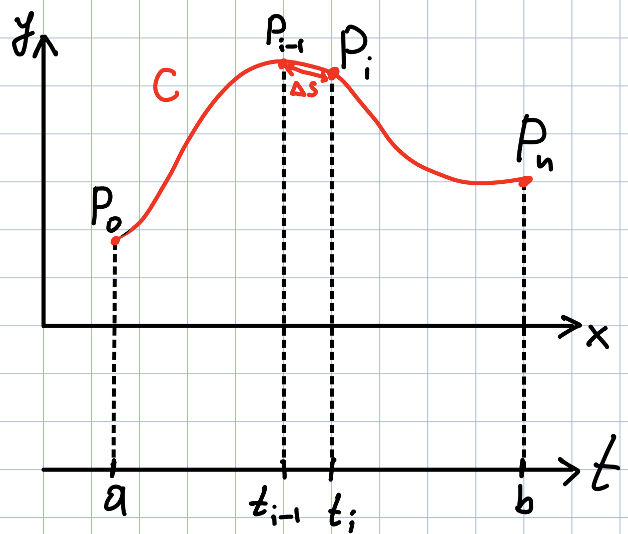
Line integrals of functions



C-plane curve $\quad x = x(t), \; y = y(t) \qquad a \le t \le b$

Line (curve) integral of $f(x,y)$ along C:

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\underbrace{x_i^*, y_i^*}_{\text{sample point in the i-th sub-arc}}) \underbrace{\Delta S_i}_{\text{length of i-th sub-arc}}$$

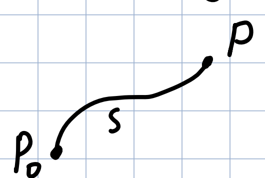
__Recall__: the length of the curve:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

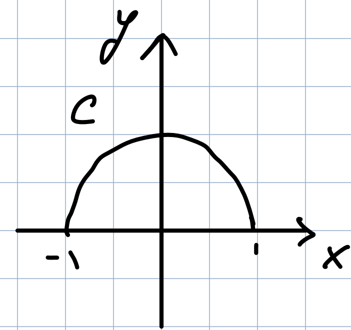
$\leftarrow$ does not depend on the parametrization of C (assuming that C is traversed exactly ones as t goes from a to b)

__Rmk__: If C is parametrised by the arc length, then $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1$



Ex: Find $\int_C (2+x^2y)\,ds$, where C is upper semi-circle $\begin{matrix} x^2+y^2=1 \\ y \geq 0 \end{matrix}$.

Sol: Parametrize C by $x=\cos t,\ y=\sin t$
$0 \leq t \leq \pi$.



$$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t \sin t)\underbrace{\sqrt{(x')^2+(y')^2}}_{\sqrt{\sin^2 t+\cos^2 t}=1}\,dt$$

$$= 2t - \frac{1}{3}\cos^3 t \Big|_0^{\pi} = 2\pi - \frac{1}{3}(-1-1) = 2\pi + \frac{2}{3}$$

Ex: Find $\int_C 2x\,ds$, where $C = C_1 \cup C_2$

C_1: arc of parabola $y=x^2$ from $(0,0)$ to $(1,1)$

C_2: line segment from $(1,1)$ to $(1,2)$.

Sol: $\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$

C_1: $x = x$, $y = x^2$, $0 \leq x \leq 1$ ← parameter

C_2: $x=1$, $y=y$, $1 \leq y \leq 2$ ← parameter

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{1+4x^2}\,dx$$
$$= \frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}$$

$$\int_{C_2} 2x\,dx = \int_1^2 2\cdot 1\sqrt{0+1}\,dy = 2y\Big|_1^2 = 2 \implies \int_C 2x\,ds = \frac{5\sqrt{5}-1}{6} + 2$$

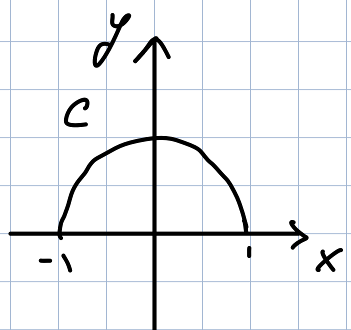
Rmk: If C is the form of a (thin) wire with linear density $\rho(x,y)$,

then $$m = \int_C \rho(x,y)\,ds \quad \text{- mass}$$

$$\bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds$$

$(\bar{x}, \bar{y})$ - center of mass.

Ex:



Shape of the wire: semicircle $\begin{matrix} x^2+y^2=1 \\ y \geq 0 \end{matrix}$ | Density: $\rho(x,y) = k(1-y)$, k const.

Find the center of mass.

Sol:

$$m = \int_C k(1-y)\,ds \ \underset{y=\sin t}{\overset{x=\cos t}{=\!=\!=}} \ \int_0^\pi k(1-\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt = \overbrace{\int_0^\pi k(1-\sin t)\,dt}^{k(t+\cos t)\big|_0^\pi} = k(\pi-2)$$

$$\bar{y} = \frac{1}{m}\int_C k(1-y)\,y\,ds \ \underset{y=\sin t}{\overset{x=\cos t}{=\!=\!=}} \ \frac{1}{m}\int_0^\pi k(1-\sin t)\sin t\,dt = \frac{1}{\pi-2}\int_0^\pi \left(\sin t - \frac{1-\cos 2t}{2}\right)dt$$

$$= \frac{1}{\pi-2}\left(-\cos t - \frac{t}{2} + \frac{1}{4}\sin 2t\Big|_0^\pi\right) = \frac{1}{\pi-2}\left(2-\frac{\pi}{2}\right) = \frac{4-\pi}{2(\pi-2)}$$

$\bar{x} = 0$ by symmetry $\Rightarrow$ $(\bar{x}, \bar{y}) = \left(0, \dfrac{4-\pi}{2(\pi-2)}\right)$ - center of mass.

≈ 0.38

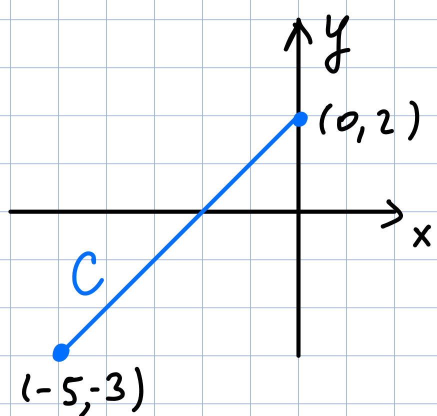
Line integrals w.r.t. x, y:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum f(x_i^*, y_i^*)\,\Delta x_i = \int_a^b f(x(t), y(t))\,\underbrace{x'(t)}\,dt$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum f(x_i^*, y_i^*)\,\Delta y_i = \int_a^b f(x(t), y(t))\,\underbrace{y'(t)}\,dt$$

Ex: Find $\int_C y^2\,dx + x\,dy := \int_C y^2\,dx + \int_C x\,dy$, where C is line segment from $(-5,-3)$ to $(0,2)$

Sol:



1) Parametrize C: $\vec{r}(t) = \underbrace{\vec{r}_0}_{\langle -5,-3\rangle}(1-t) + \underbrace{\vec{r}_1}_{\langle 0,2\rangle}\, t$

$0 \le t \le 1$

or $x = -5 + 5t \qquad y = -3 - (-3)t + 2t = -3 + 5t$

$\Rightarrow \quad x'(t) = 5 \quad , y'(t) = 5$

$$\int_C y^2\,dx + \int_C x\,dy = \int_0^1 \left((-3+5t)^2 \cdot 5 + (-5+5t)\cdot 5\right) dt = 5\int_0^1 (25t^2 - 25t + 4)\,dt$$

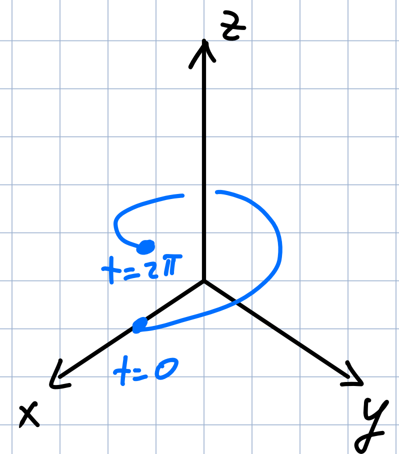
$$= 5\left(\tfrac{25}{3}t^3 - \tfrac{25}{2}t^2 + 4t\Big|_0^1\right) = 5\left(\tfrac{25}{3} - \tfrac{25}{2} + 4\right) = -\tfrac{5}{6}$$

Rmk: Note that if C were from $(0,2)$ to $(-5,-3)$ (i.e. opposite orientation) we would get $+\frac{5}{6}$

Generally: $\displaystyle \underbrace{\int_{-C}}_{C \text{ travelled in opposite direction}} f(x,y)\,ds = -\int_C f(x,y)\,ds$

• In space: $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Ex: Find $\displaystyle\int_C y \sin z\,ds$, where C is circular helix $\begin{array}{l} x = \cos t \\ y = \sin t \\ z = t \end{array}$, $0 \le t \le 2\pi$



Sol: $\displaystyle\int_C y \sin z\,ds = \int_0^{2\pi} \sin t \sin t \sqrt{\underbrace{(x')^2 + (y')^2 + (z')^2}_{\sin^2 t + \cos^2 t + 1}}\,dt$

$$= \int_0^{2\pi} \frac{1-\cos 2t}{2} \sqrt{2}\,dt = \sqrt{2}\left(\frac{t}{2} - \frac{\sin 2t}{4}\Big|_{t=0}^{t=2\pi}\right)$$

$$= \pi\sqrt{2}$$

Rmk: $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

$$= \int_C \Big(P(x(t),y(t),z(t))\,x'(t) + Q(x(t),y(t),z(t))\,y'(t) + R(x(t),y(t),z(t))\,z'(t) \Big)\,dt$$